LINEAR BALLS AND THE MULTIPLICITY CONJECTURE

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Abstract. A linear ball is a simplicial complex whose geometric realization is homeomorphic to a ball and whose Stanley–Reisner ring has a linear resolution. It turns out that the Stanley–Reisner ring of the sphere which is the boundary complex of a linear ball satisfies the multiplicity conjecture. A class of shellable spheres arising naturally from commutative algebra whose Stanley–Reisner rings satisfy the multiplicity conjecture will be presented.

Introduction

The multiplicity conjecture due to Herzog, Huneke and Srinivasan is one of the most attractive conjectures lying between combinatorics and commutative algebra. First, we recall what the multiplicity conjecture says.

Let $R = \sum_{i=0}^{\infty} R_i$ be a homogeneous Cohen–Macaulay algebra over a field $R_0 = K$ of dimension d with embedded dimension $n = \dim_K R_1$ and write $R = S/I$, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables over K and I is a graded ideal of S. Let $H(R, i) = \dim_K R_i$, $i = 0, 1, 2, \ldots$, denote the Hilbert function of R and $F(R, \lambda) = \sum_{i=0}^{\infty} H(R, i)\lambda^i$ the Hilbert series of R. It is known that $F(R, \lambda)$ is a rational function of λ of the form

$$F(R, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_\ell\lambda^\ell}{(1 - \lambda)^d},$$

with each $h_i > 0$. The multiplicity $e(R)$ of R is

$$e(R) = h_0 + h_1 + \cdots + h_\ell.$$

Now, we consider the graded minimal free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow R \longrightarrow 0$$

of R over S, where $F_i = \bigoplus S(-j)^{\beta_{i,j}}$ with $\beta_{i,j} \geq 0$. Let

$$m_i = \min\{j : \beta_{i,j} \neq 0\}, \qquad M_i = \max\{j : \beta_{i,j} \neq 0\}.$$

The multiplicity conjecture due to Herzog, Huneke and Srinivasan says that

$$\frac{\prod_{i=1}^{p} m_i}{p!} \leq e(R) \leq \frac{\prod_{i=1}^{p} M_i}{p!}.$$

A nice survey of the multiplicity conjecture and the record of past results in different cases of the conjecture can be found in [13]. For more recent results one may look into [15], [16], [17].

In the present article we discuss the problem of finding a natural class of spheres whose Stanley–Reisner rings satisfy the multiplicity conjecture.

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ of dimension $d-1$ and $K[\Delta] = S/I_\Delta$, where $S = K[x_1, \ldots, x_n]$, its Stanley–Reisner ring. Suppose that Δ is a ball, i.e., the geometric realization $|\Delta|$ is a ball. Let $\partial\Delta$ denote the boundary complex of Δ and suppose that each vertex of Δ belongs to $\partial\Delta$. Thus $\partial\Delta$ is a sphere, i.e., the geometric realization $|\partial\Delta|$ is a sphere, of dimension $d-2$ on $[n]$. Each face of $\partial\Delta$ is called a boundary face of Δ and each face of $\Delta \setminus \partial\Delta$ is called an inside face of Δ. Let $m-1$ denote the smallest dimension of a nonface of Δ and suppose that $2 \leq m \leq [(d+1)/2]$. It turns out (Theorem 1.2) that the sphere $\partial\Delta$ satisfies the multiplicity conjecture with assuming the hypothesis that

(A1) Δ has a minimal inside face of dimension $d-m$ and has no minimal inside face of dimension less than $m-1$;
(A2) the h-vector of $\partial\Delta$ is unimodal.

A linear ball is a ball whose Stanley–Reisner ring has a linear resolution. It is shown that the sphere which is the boundary complex of a linear ball satisfies (A1) and (A2). In particular the Stanley–Reisner ring of the sphere which is the boundary complex of a linear ball satisfies the multiplicity conjecture (Corollary 1.4).

A class of shellable spheres satisfying (A1) and (A2) arises from determinantal ideals. Let $X = (X_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ be an $m \times n$ matrix of indeterminates, where $m \leq n$. Write τ for the lexicographic order of the polynomial ring $K[X] = K[\{X_{ij}\}_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}]$ induced by the ordering of the variables

$$X_{11} \geq X_{12} \geq \cdots \geq X_{1n} \geq X_{21} \geq \cdots \geq X_{2n} \geq \cdots \geq X_{m1} \geq \cdots \geq X_{mn}.$$

Let I_r denote the ideal of $K[X]$ generated by all $(r+1)\times(r+1)$ minors of X, where $1 \leq r \leq m-1$. In particular I_{m-1} is the ideal of $K[X]$ generated by all maximal minors of X. It is known that the initial ideal I_r^* of I_r with respect to τ is generated by squarefree monomials. Let Δ_r denote the simplicial complex whose Stanley–Reisner ideal coincides with I_r^*. Theorem 2.4 says that, for each $1 \leq r \leq m-1$, the simplicial complex Δ_r is a shellable ball satisfying (A1) and (A2). Moreover Δ_r is a linear ball if and only if $r = m-1$ (Corollary 2.5).

One of the natural classes of shellable linear balls arises from the polarization of a power of the graded maximal ideal. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of $S = K[x_1, \ldots, x_n]$. Each power $\mathfrak{m}^t$ of $\mathfrak{m}$ has a linear resolution. Let Δ be the simplicial complex whose Stanley–Reisner ideal coincides with the polarization of $\mathfrak{m}^t$. It is shown (Theorem 3.1) that Δ is a shellable linear ball for $t \geq 0$ and hence it satisfies the multiplicity conjecture.

1. The Multiplicity Conjecture

First, we recall fundamental material on Stanley–Reisner ideals and rings of simplicial complexes. We refer the reader to [1], [8], [18] for further information. Let $[n] = \{1, \ldots, n\}$ be the vertex set and Δ a simplicial complex on $[n]$. Thus Δ is a collection of subsets of $[n]$ such that

(i) $\{i\} \in \Delta$ for all $i \in [n]$, and
(ii) if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$.

Each element $F \in \Delta$ is called a *face* of Δ. The dimension of a face F is $|F| - 1$. Let $d = \max\{|F| : F \in \Delta\}$ and define the dimension of Δ to be $\dim \Delta = d - 1$. A *nonface* of Δ is a subset F of $[n]$ with $F \not\in \Delta$.

Let $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i. Thus in particular $f_0 = n$. The sequence $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Letting $f_{-1} = 1$, we define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. For each subset $F \subset [n]$, we set

$$x_F = \prod_{i \in F} x_i.$$

The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \not\in \Delta$. In other words,

$$I_\Delta = (x_F : F \not\in \Delta).$$

The quotient ring $K[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of Δ. It follows that the Hilbert series of $K[\Delta]$ is

$$F(K[\Delta], \lambda) = (h_0 + h_1\lambda + \cdots + h_d\lambda^d)/(1-\lambda)^d,$$

where $(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ. Thus in particular the multiplicity of $K[\Delta]$ is $\sum_{i=0}^{d} h_i$ $(= f_{d-1})$.

We say that Δ is Cohen–Macaulay (resp. Gorenstein) over K if $K[\Delta]$ is Cohen–Macaulay (resp. Gorenstein). If the geometric realization $|\Delta|$ of Δ is homeomorphic to a ball, then Δ is Cohen–Macaulay over an arbitrary field. If the geometric realization $|\Delta|$ of Δ is homeomorphic to a sphere, then Δ is Gorenstein over an arbitrary field.

Now, let Δ be a simplicial complex on $[n]$ of dimension $d-1$ whose geometric realization $|\Delta|$ is homeomorphic to a manifold. The *boundary complex* $\partial\Delta$ of Δ consists of those faces F of Δ with the property that there is a $(d-2)$-dimensional face F' of Δ with $F \subset F'$ such that F' is contained in exactly one $(d-1)$-dimensional face of Δ. Each face of $\partial\Delta$ is called a *boundary face* and each face of $\Delta \setminus \partial\Delta$ is called an *inside face* of Δ. In particular if Δ is a ball, i.e., $|\Delta|$ is homeomorphic to a ball, of dimension $d-1$, then $\partial\Delta$ is a sphere, i.e., $|\partial\Delta|$ is homeomorphic to a sphere, of dimension $d-2$.

Theorem 1.1 (Hochster)**.** *Let Δ be a Cohen–Macaulay complex over a field K of dimension $d-1$ whose geometric realization $|\Delta|$ is a manifold with a nonempty boundary complex $\partial\Delta$, and let ω_Δ be the canonical ideal of $K[\Delta]$. Write J for the ideal of $K[\Delta]$ generated by those monomials $\overline{x_F}$ with $F \in \Delta \setminus \partial\Delta$. Then the following conditions are equivalent:*

(a) *$\omega_\Delta \cong J$ as a $\mathbb{Z}^n$-graded $K[\Delta]$-module;*
(b) *$\partial\Delta$ is a Gorenstein complex over K.*

If the equivalent conditions hold, then $K[\partial\Delta] \cong K[\Delta]/\omega_\Delta$.

Let Δ be a simplicial complex on $[n]$ of dimension $d-1$ whose geometric realization $|\Delta|$ is a ball and $\partial\Delta$ its boundary complex. Assume that every vertex of Δ belongs to $\partial\Delta$. Thus $\partial\Delta$ is a simplicial complex on $[n]$ of dimension $d-2$ whose geometric realization $|\partial\Delta|$ is a sphere. Since $\partial\Delta$ is Gorenstein, it follows that

(P1) The h-vector $h(\partial\Delta) = (h'_0, h'_1, \dots, h'_{d-1})$ of $\partial\Delta$ is symmetric i.e. $h'_i = h'_{d-1-i}$ for all $i = 0, \dots, d-1$; see [1, Theorem 5.4.2, Theorem 5.6.2].

(P2) The minimal free resolution of the Stanley–Reisner ring of $\partial\Delta$ is symmetric ([7, Corollary 21.16]), i.e. if

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I_{\partial\Delta} \longrightarrow 0$$

is the minimal free resolution of the ring $S/I_{\partial\Delta}$, where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, $i = 0, \dots, p$, $p = n-(d-1)$ and $F_0 = S$, then we have $\beta_{i,j} = \beta_{p-i,n-j}$ for all $i = 0, \dots, p$. In particular, $M_i = n - m_{p-i}$ where $M_i = \max\{j\colon\ \beta_{i,j} \neq 0\}$ and $m_i = \min\{j\colon\ \beta_{i,j} \neq 0\}$.

(P3) The canonical ideal ω_Δ of the Stanley–Reisner ring $K[\Delta] = S/I_\Delta$ is generated by the monomials $\overline{x_F}$, $F \in \Delta \setminus \partial\Delta$ (see Theorem 1.1).

In addition,

(F1) Let

$$0 \longrightarrow F'_{n-d} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow F'_0 \longrightarrow S/I_\Delta \longrightarrow 0$$

be the minimal free resolution of S/I_Δ with $F'_i = \bigoplus_j S(-j)^{\beta'_{i,j}}$. Then the generators of the canonical module ω_Δ of $K[\Delta]$ are of degrees $n-j$ with $\beta'_{n-d,j} \neq 0$ (see [1, Corollary 3.3.9]).

(F2) One has $m_1 < m_2 < \cdots < m_{n-d+1}$.

Now, let $m-1$ denote the smallest dimension of the nonfaces of Δ. In other words, m is the smallest degree of monomials belonging to $G(I_\Delta)$, the minimal system of monomial generators of I_Δ. We will assume that $2 \leq m \leq [(d+1)/2]$. Our goal is to show that the Stanley–Reisner ring $K[\partial\Delta] = S/I_{\partial\Delta}$ satisfies the multiplicity conjecture under the following hypothesis (Theorem 1.2):

(A1) Δ has a minimal (under inclusion) inside face of dimension $d-m$ and has no minimal inside face of dimension less than $m-1$;

(A2) The h-vector of the boundary complex $\partial\Delta$ is unimodal.

(In general, we say that a finite sequence of real numbers $a_0, \dots, a_t$ is *unimodal* if

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_t$$

for some $0 \leq j \leq t$.)

Now, we wish to understand the minimal and maximal shifts given by m_i and M_i respectively of the minimal free resolution

$$\mathcal{F}_{\partial\Delta} : 0 \longrightarrow F_{n-d+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow S/I_{\partial\Delta} \longrightarrow 0$$

of $S/I_{\partial\Delta}$ where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, to calculate the lower and upper bounds of the multiplicity of $S/I_{\partial\Delta}$. First, we consider the minimal free resolution

$$\mathcal{F}_\Delta : 0 \longrightarrow F'_{n-d} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow S \longrightarrow S/I_\Delta \longrightarrow 0$$

of S/I_Δ where $F'_i = \bigoplus_j S(-j)^{\beta'_{i,j}}$. Let m'_i and M'_i denote the minimal and maximal shifts of the minimal free resolution $\mathcal{F}_\Delta$. Since m is the minimum of the degree of generators of I_Δ, one has $m'_1 = m$. By the assumption (A1) on Δ, there exists a minimal inside face of Δ of dimension $d-m$, hence by Theorem 1.1, it follows that the canonical ideal ω_Δ of Δ has a generator of degree $d-m+1$. Therefore $\beta'_{n-d,n-(d-m+1)} \neq 0$, by (F1). As we have $m'_1 = m$ and $m'_{n-d} \leq m+n-d-1$, we get $m'_i = m+i-1$ for $i = 1, \ldots, n-d$, by (F2).

We claim that the minimal shifts in the minimal free resolution $\mathcal{F}_{\partial\Delta}$ of $S/(I_{\partial\Delta})$ are given by $m_i = m+i-1$ for $i = 1, \ldots, n-d$ and $m_{n-d+1} = n$. Indeed, by assumption (A1), we have that the canonical ideal ω_Δ has no generator of degree less than m. Hence the S-module $I_{\partial\Delta}/I_\Delta$ has no generator of degree less than m (Theorem 1.1). From the following short exact sequence

$$0 \longrightarrow I_\Delta \longrightarrow I_{\partial\Delta} \longrightarrow I_{\partial\Delta}/I_\Delta \longrightarrow 0,$$

we get the following long exact sequence

$$\begin{aligned} &\cdots \longrightarrow \mathrm{Tor}_{i+1}(I_{\partial\Delta}/I_\Delta, K) \\ \longrightarrow \mathrm{Tor}_i(I_\Delta, K) \longrightarrow \mathrm{Tor}_i(I_{\partial\Delta}, K) \longrightarrow &\mathrm{Tor}_i(I_{\partial\Delta}/I_\Delta, K) \longrightarrow \cdots \end{aligned}$$

Now, as $\mathrm{Tor}_i(I_\Delta, K)_{i+t} = 0$ and $\mathrm{Tor}_i(I_{\partial\Delta}/I_\Delta, K)_{i+t} = 0$ for $t \leq m-1$ and $i = 1, \ldots, n-d$, from the above long exact sequence we get $\mathrm{Tor}_i(I_{\partial\Delta}, K)_{i+t} = 0$ for $t \leq m-1$ and $i = 1, \ldots, n-d$. Also as $\mathrm{Tor}_{i+1}(I_{\partial\Delta}/I_\Delta, K)_{i+1+m-1} = 0$ and $\mathrm{Tor}_i(I_\Delta, K)_{i+m} \neq 0$, we get $\mathrm{Tor}_i(I_{\partial\Delta}, K)_{i+m} \neq 0$, $i = 1, \ldots n-d$. From here it follows that $m_i = m+i-1$ for $i = 1, \ldots, n-d$. Since $S/I_{\partial\Delta}$ is Gorenstein and $m_0 = M_0 = 0$, we have $m_{n-d+1} = M_{n-d+1} = n$ by Property (P2).

Now, we need to determine the maximal shifts M_i for $i = 1, \ldots, n-d$ in the minimal free resolution $\mathcal{F}_{\partial\Delta}$ of $S/I_{\partial\Delta}$. Again, as $S/I_{\partial\Delta}$ is Gorenstein, by Property (P2) we have $M_i = n - m_{n-d+1-i} = n - (m+n-d+1-i-1) = d-m+i$ for $i = 1, \ldots, n-d$.

Hence, we have now

$$\begin{aligned} L &= \prod_{i=1}^{n-d+1} \frac{m_i}{(n-d+1)!} = \frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!} \quad \text{and} \\ U &= \prod_{i=1}^{n-d+1} \frac{M_i}{(n-d+1)!} = \frac{n\prod_{i=1}^{n-d}(d-m+i)}{(n-d+1)!}. \end{aligned}$$

Next, our goal is to estimate the multiplicity $e(S/I_{\partial\Delta})$ of the ring $S/I_{\partial\Delta}$. Let $h'_0, \ldots, h'_{d-1}$ denotes the h-vector of the ring $S/I_{\partial\Delta}$. As the ring $S/I_{\partial\Delta}$ is Cohen-Macaulay, and m is the minimum of the degree of the generators of $I_{\partial\Delta}$, we have $h'_i = h'_{d-1-i} = \binom{n-d+1+i-1}{i} = \binom{n-d+i}{i}$ for $i = 0, \ldots, m-1$. From assumption (A2) and property (P1) we have that the h-vector is symmetric and unimodal, therefore we conclude that $h'_i \geq \binom{n-d+m-1}{m-1}$ for $i = m, \ldots, d-(m+1)$.

Hence

$$e(S/I_{\partial\Delta}) = \sum_{i=1}^{d-1} h_i \geq 2\sum_{i=0}^{m-1}\binom{n-d+i}{i} + (d-2m)\binom{n-d+m-1}{m-1}.$$

Theorem 1.2. *Let Δ be a ball and $\partial\Delta$ be its boundary complex. Suppose that the sphere $\partial\Delta$ satisfies the assumptions (A1) and (A2). Then the Stanley-Reisner ring $S/\partial\Delta$ satisfies the multiplicity conjecture i.e.*

$$L \leq e(S/I_{\partial\Delta}) \leq U.$$

For the proof of the theorem, we need to first define cyclic polytopes. Let $C(n, d-1)$ denote the convex hull of any n distinct points in $\mathbb{R}^{d-1}$ on the curve $\{(t, t^2, \ldots, t^{d-1}) \in \mathbb{R}^{d-1}, t \in \mathbb{R}\}$. The polytope $C(n, d-1)$ is called the cyclic polytope of dimension $d-1$. It is known that $C(n, d-1)$ is simplicial (i.e., every proper face is a simplex), and so the boundary of $C(n, d-1)$ defines a simplicial complex which we denote by $\partial C(n, d-1)$ such that $|\partial C(n, d-1)|$ is a sphere of dimension $d-2$. Let $(h_0^*, h_1^*, \ldots, h_{d-1}^*)$ denote the h-vector of $\partial C(n, d-1)$. Then

$$h_i^* = h_{d-1-i}^* = \binom{n-d+i}{i} \quad \text{for} \quad i = 1, \ldots, \lfloor \frac{d-1}{2} \rfloor,$$

(see [18, Section 3]). Let $e(\partial C(n, d-1)) = \sum h_i^*$ denotes the multiplicity of the Stanley-Reisner ring of the boundary complex $\partial C(n, d-1)$. Notice that we have $h_i' \leq h_i^*$, hence

$$e(S/I_{\partial\Delta}) \leq e\big(\partial C(n, d-1)\big). \tag{1}$$

In [20], the minimal free resolution of the $\partial C(n, d-1)$ is computed. We have the following [20, Theorem 3.2]: If $d-1 \geq 2$ is even, then the maximal shifts M_i^* in the minimal free resolution of $\partial C(n, d-1)$ are given by

$$M_i^* = \frac{d-1}{2} + i \quad \text{for} \quad i = 1, \ldots, n-d \quad \text{and} \quad M_{n-d+1}^* = n \tag{2}$$

and if $d-1 \geq 3$ is odd, then the maximal shifts M_i^* are as follows:

$$M_i^* = \lfloor \frac{d-1}{2} \rfloor + i + 1 \quad \text{for} \quad i = 1, \ldots, n-d \quad \text{and} \quad M_{n-d+1}^* = n. \tag{3}$$

Even though the following Lemma 1.3 follows from [10, Theorem 1.2], we want to give a direct computational proof.

Lemma 1.3. *We have*

$$e\big(\partial C(n, d-1)\big) \leq \frac{\prod_{i=1}^{n-d+1} M_i^*}{(n-d+1)!}. \tag{4}$$

Proof. Let $U = \frac{\prod_{i=1}^{n-d+1} M_i^*}{(n-d+1)!}$. Let first $d-1 \geq 2$ is even. Then

$$U = \frac{n(\frac{d}{2} + \frac{1}{2})(\frac{d}{2} + \frac{3}{2}) \cdots (n - \frac{d}{2} - \frac{1}{2})}{(n-d+1)!}.$$

We have the multiplicity

$$
\begin{aligned}
& e\big(\partial C(n,d-1)\big) = \sum_{i=0}^{d-1} h^* \\
= \;& 2\Big[\binom{n-d+0}{0} + \cdots + \binom{(n-d)+d/2-3/2}{d/2-3/2}\Big] + \binom{(n-d)+d/2-1/2}{d/2-1/2} \\
= \;& 2\binom{n-d/2-1/2}{d/2-3/2} + \binom{n-d/2-1/2}{d/2-3/2} \\
= \;& \frac{2(n-d/2-1/2)\cdots(d/2-1/2)}{(n-d+1)!} + \frac{(n-d/2-1/2)\cdots(d/2+1/2)}{(n-d)!} \\
= \;& \frac{(n-d/2-1/2)\cdots(d/2+1/2)}{(n-d+1)!}(d-1+n-d+1) \\
= \;& U.
\end{aligned}
$$

Now let $d-1 \geq 3$ be odd. Then

$$
U = \frac{n\big(\frac{d}{2}+1\big)\cdots\big(\frac{d}{2}+(n-d)\big)}{(n-d+1)!}.
$$

And the multiplicity is given by

$$
\begin{aligned}
e\big(\partial C(n,d-1)\big) &= \sum_{i=0}^{d-1} h^* \\
&= 2\Big[\binom{n-d+0}{0} + \binom{n-d+1}{1} + \cdots + \binom{n-d+d/2-1}{d/2-1}\Big] \\
&= 2\binom{n-d/2}{d/2-1} \\
&= 2\frac{(n-d/2)\cdots(d/2+1)(d/2)}{(n-d+1)!}.
\end{aligned}
$$

We see that $e\big(\partial C(n,d-1)\big) \leq U$ if and only if $d \leq n$ which is true. $\square$

Proof of Theorem 1.2. Since $m \leq [(d+1)/2]$, we have $M_i^* \leq M_i$ both when d is odd and even. Hence, by Equation (1) and Equation (4), we get

$$
e(S/I_{\partial\Delta}) \leq \frac{\prod_{i=1}^{n-d+1} M_i}{(n-d+1)!}. \tag{5}
$$

It remains to show that $e(S/I_{\partial\Delta}) \geq L$. Since

$$
e(S/I_{\partial\Delta}) \geq 2\sum_{i=0}^{m-1}\binom{n-d+i}{i} + (d-2m)\binom{n-d+m-1}{m-1},
$$

it is enough to show that

$$2\sum_{i=0}^{m-1}\binom{n-d+i}{i}+(d-2m)\binom{n-d+m-1}{m-1}\geq \frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!}$$

which is to prove

$$2\binom{n-d+m}{m-1}+(d-2m)\binom{n-d+m-1}{m-1}\geq \frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!}.$$

We need to show

$$\begin{aligned}2(n-d+m)\cdots(m+1)(m)+(d-2m)(n-d+m-1)\cdots(m+1)(m)(n-d+1)\\ \geq n(m)(m+1)\cdots(m+n-d-1)\end{aligned}$$

which further amounts to prove that $2(n-d+m)+(d-2m)(n-d+1)\geq n$. Notice that it is enough to show that $2(n-d+m)+(d-2m)\geq n$ which is true as $n>d$. □

Corollary 1.4. *Let Δ be a linear ball. Then the simplicial sphere $\partial\Delta$ satisfies the multiplicity conjecture.*

Proof. We only need to show that the assumptions (A1) and (A2) are satisfied in this case. Since S/I_Δ has a linear resolution, the minimal and maximal shifts in the minimal free resolution of S/I_Δ are given by $m_i'=M_i'=m+i-1$ for $i=1,\ldots,n-d$. Hence Δ has inside faces only of dimension $n-(m+n-d-1)-1=d-m$, by fact (F1) and Theorem 1.1. Also, there is no inside face of dimension less than $m-1$ since $d-m\geq m-1$. Hence the assumption (A1) is satisfied. We now show that the h vector $(h_0',\ldots,h_{d-1}')$ of $S/I_{\partial\Delta}$ is unimodal. As the Stanley-Reisner ideal I_Δ has linear resolution and $S=K[\Delta]=S/I_\Delta$ is Cohen-Macaulay, we get that the h-vector $(h_0,\ldots,h_d)$ of S/I_Δ is given by $h_i=\binom{n-d+(i-1)}{i}$ for $i=0,\ldots,m-1$ and $h_i=0$ for $i\geq m$.

Now the h-vector of $S/I_{\partial\Delta}$ is equal to (see [18, p. 137]) :

$$(h_0-h_d,h_0+h_1-h_d-h_{d-1},\ldots,h_0+\cdots+h_{d-1}-h_d-\cdots-h_1).$$

Hence the h-vector of $S/I_{\partial\Delta}$ is given by

$$h_i'=\begin{cases}\binom{n-d+i}{i} & \text{for } i=0,\ldots,m-2;\\[2ex] \binom{n-d+m-1}{m-1} & \text{for } i=m-1,\ldots,d-m;\\[2ex] \binom{n-d+(d-1-i)}{d-1-i} & \text{for } i=d-m+1,\ldots,d-1.\end{cases}$$

Hence the assumption (A2) also holds. □

2. Determinantal Ideals

In this section, we study simplicial complexes arising from determinantal ideals. It is known that these simplicial complexes are shellable. We prove that the geometric realization of these simplicial complexes are balls and these balls are linear only in the case of the ideal of maximal minors. We show that the boundary complexes of these simplicial complexes satisfy the multiplicity conjecture.

Let $X = (X_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, n$, $m \leq n$ be an $m \times n$ matrix of indeterminates. We denote by $[a_1, \dots, a_r \,|\, b_1, \dots, b_r]$, the minor $\det(X_{a_i b_j})$ of X where $i, j = 1, \dots, r$. Further we define

$$[a_1, \dots, a_r \,|\, b_1, \dots, b_r] \leq [a_1^{'}, \dots, a_s^{'} \,|\, b_1{'}, \dots, b_s^{'}],$$

if $r \geq s$ and $a_i \leq a_i^{'}$, $b_i \leq b_i^{'}$ for $i = 1, \dots, s$. Let $\Delta(X)$ denote the poset of minors of X. For $\sigma = [a_1, \dots, a_r \,|\, b_1, \dots, b_r] \in \Delta(X)$, we denote by I_σ the ideal generated by all minors $\gamma \not\geq \sigma$. We call such ideals determinantal ideals. Notice that for $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, $r \leq m-1$, the ideal I_σ is the ideal generated by all $(r+1) \times (r+1)$ minors of X. For $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, $r \leq m-1$, we denote the ideal I_σ by I_r. Note that the ideal I_{m-1} is generated by all maximal minors of X.

Let the symbol τ denote the lexicographic term order on the polynomial ring $S = K[X] = K[X_{ij},\ i = 1, \dots, m,\ j = 1, \dots, n]$ induced by the variable order

$$X_{11} \geq X_{12} \geq \cdots \geq X_{1m} \geq X_{21} \geq X_{22} \cdots \geq X_{2m} \geq X_{n1} \geq X_{n2} \geq \cdots \geq X_{mn}.$$

Notice that under the monomial order τ, the initial monomial of any minor of X is the product of the elements of its main diagonal. Such a monomial order is called diagonal order. In [11], it is shown that the generators of I_σ form a Gröbner basis and hence I_σ^* of I_σ with respect to the monomial order τ, is generated by squarefree monomials. In other words, $K[X]/I_\sigma^*$ may be viewed as a Stanley-Reisner ring of a certain simplicial complex Δ_σ. For $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, $r \leq m-1$, we denote the simplicial complex Δ_σ by Δ_r.

We show in Theorem 2.4 that for any $\sigma = [a_1, \dots, a_r \,|\, b_1, \dots, b_r] \in \Delta(X)$, the geometric realization $|\Delta_\sigma|$ of the simplicial complex Δ_σ is a shellable ball. By Theorem 2.4 and Corollary 2.5 together, it follows that the geometric realization $|\Delta_{m-1}|$ of Δ_{m-1} is in fact a shellable linear ball.

According to [11], the facets of simplicial complex Δ_σ can be described as follows: its vertex set is the set of coordinate points $V = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. We define a partial order on V by setting $(i, j) \leq (i^{'}, j^{'})$ if $i \geq i^{'}$ and $j \leq j^{'}$. A maximal chain in V will be called a *path*.

Theorem 2.1. [11, Theorem 3.3] *Let $\sigma = [a_1, \dots, a_r \,|\, b_1, \dots, b_r]$, and let $P_i = (a_i, n)$ and $Q_i = (m, b_i)$ for $i = 1, \cdots, r$. Then the facets of Δ_σ are the non-intersecting paths from P_i to Q_i, that is, subsets $C_1 \cup C_2 \cup \cdots \cup C_r$ of V where each C_i is a path with end points P_i and Q_i and where $C_i \cap C_j = \emptyset$ for all $i \neq j$.*

We denote the set of facets of Δ_σ by $\mathcal{F}(\Delta_\sigma)$. The complex Δ_σ has a natural partial order on the set of facets which we recall from [11, Theorem 4.9]: Let F_1 and F_2 be two facets of Δ_σ. We write $F_1 = \bigcup_{i=1}^r C_i$ and $F_2 = \bigcup_{i=1}^r D_i$ as unions of non-intersecting paths with end points P_i and Q_i. We say that $F_2 \geq F_1$, if D_i is contained

in the upper right side of C_i for all $i = 1, \dots, r$, that is, if for each $(x, y) \in D_i$ there is some $(u, v) \in C_i$ such that $u \le x$ and $v \le y$, where $i = 1, \dots, r$. This is a partial order on the facets of Δ_σ, and this partial order extended to any linear order gives us a shelling. We fix a linear order and let Σ denotes the corresponding shelling. From [4, Corollary 5.18], we have $\dim(S/I^*_\sigma) = r(m+n+1) - \sum_{i=1}^r (a_i + b_i)$.

Before stating the next theorem, we define the notion of a *corner* of a path. Let C be a path in V. A point $(i, j) \in C$ will be called a *corner* of C, if $(i-1, j)$ and $(i, j-1)$ belong to C. Let F be a facet of Δ_σ, then we denote by $\mathcal{C}(F)$, the set of corners of the paths in F, and we define $c(F) = |\mathcal{C}(F)|$.

For the proof of Theorem 2.4, we need the following lemma from algebraic topology:

Lemma 2.2. *Let E_1 be a simplicial complex whose geometric realization $|E_1|$ is a ball of dimension d, and let E_2 be a simplex of dimension d. Let the intersection $E_1 \cap E_2 = \langle G_1, \dots, G_r \rangle \neq \emptyset$, where $G_1, \dots, G_r$ are facets of the boundary complexes ∂E_i of E_i, $i = 1, 2$ and $\langle G_1, \dots, G_r \rangle$ is a proper subset of ∂E_2. Then the geometric realization $|E_1 \cup E_2|$ of $E_1 \cup E_2$ is again a ball.*

The following lemma follows from the proof of [2, Theorem 2.4].

Lemma 2.3. *Let $\Delta_\sigma = \langle F_1, \dots, F_t \rangle$ be the simplicial complex with Stanley-Reisner ideal I_σ where $F_1, \dots, F_t$ is the shelling order Σ. Let $\Delta_i = \langle F_1, \dots, F_i \rangle$ and let $G = F_k \setminus \{v\}$ for some $v \in F_k$, $k \le i$. Then $G \subset F_\ell$ for some $\ell < k$ if and only if $v \in \mathcal{C}(F_k)$. If the equivalent conditions hold then F_ℓ is uniquely determined.*

Theorem 2.4. *For any $\sigma = [a_1, \dots, a_r \,|\, b_1, \dots, b_r] \in \Delta(X)$, the geometric realization $|\Delta_\sigma|$ of the simplicial complex Δ_σ is a shellable ball of dimension $r(m+n+1) - \sum_{i=1}^r (a_i + \ b_i) - 1$.*

Proof. The fact that the dimension of the simplicial complex Δ_σ is $r(m+n+1) - \sum_{i=1}^r (a_i + \ b_i) - 1$ follows from [4, Corollary 5.18]. Let $\Delta_\sigma = \langle F_1, \dots, F_t \rangle$ where $F_1, \dots, F_t$ is the shelling order Σ. Let $\Delta_i = \langle F_1, \dots, F_i \rangle$. We prove that $|\Delta_i|$ is a ball by induction on i. Assume that $|\Delta_{i-1}|$ is a ball, we will show that $|\Delta_i|$ is a ball. We have $\Delta_i = \Delta_{i-1} \cup \langle F_i \rangle$, let $\Delta_{i-1} \cap \langle F_i \rangle = \langle G_1, \dots, G_r \rangle$. Notice that G_j are codimension one faces of Δ_{i-1} as Δ_σ is shellable. By Lemma 2.2, we notice that $|\Delta_i|$ is a ball (assuming that $|\Delta_{i-1}|$ is a ball), if the following two conditions are satisfied:

(1) Each G_j is a subset of exactly one F_k for $k \le i-1$, which in turn implies that $G_j \in \partial \Delta_{i-1}$,
(2) $G_1, \dots, G_r$ is a proper subset of the boundary complex $\partial \langle F_i \rangle$ of $\langle F_i \rangle$.

The first condition follows from Lemma 2.3. For the second condition, we define $G_v = F_i \setminus \{v\}$ where $v \notin \mathcal{C}(F_i)$ (Notice that such a v exists as not all points in F_i are corner points of F_i). Then again from Lemma 2.3, there exists no F_j, $j \le i-1$ such that $G_v = F_j \cap F_i$. Hence $G_v \subset \partial \langle F_i \rangle$ and $G_v \neq G_j$ for $j = 1, \dots, r$. □

An ideal $I \subset S$ generated in degree d is said to have a linear resolution if in the minimal free resolution of I, one has the maximal shifts $M_i = d + i$ for all i. It is known that the ideal I_{m-1} generated by the maximal minors of matrix X has a linear

resolution. In fact, the Eagon–Northcott complex gives a minimal free resolution for I_{m-1}, see [4, Theorem 2.16]. We have the following :

Corollary 2.5. *Let Δ_r be the simplicial complex with the Stanley-Reisner Ideal I_r^*. Then $|\Delta_r|$ is a linear ball if and only if $r = m-1$.*

Proof. First we show that $|\Delta_{m-1}|$ is a linear ball i.e. we show that the Stanley Reisner ideal I_{m-1}^* has a linear resolution. As stated before, we know that the ideal I_{m-1} has a linear resolution. Moreover, the ring S/I_{m-1} is Cohen-Macaulay, see [4, Theorem 2.8]. Now as Δ_{m-1} is shellable, the ring S/I_{m-1}^* is also Cohen-Macaulay. From here it follows, that the Stanley-Reisner ideal I_{m-1}^* also has a linear resolution. Indeed, note that S/I_{m-1} and S/I_{m-1}^* have the same Hilbert function. Let $\dim S/I_{m-1} = \dim S/I_{m-1}^* = d$. Let $y_1, \ldots, y_d$ and $y_1', \ldots, y_d'$ be the maximal regular sequences of linear forms in S/I_{m-1} and in S/I_{m-1}^*, respectively. Then $\overline{S}/\overline{I_{m-1}}$ is zero dimensional $\big($here $\bar{\ }$ denotes modulo the sequence $(y_1, \ldots, y_d)\big)$ and has a linear resolution. This is only possible if $\overline{I_{m-1}}$ is a power of the maximal ideal of $\overline{S}$. Now the zero dimensional ring $\overline{S}/\overline{I_{m-1}^*}$ $\big($here $\bar{\ }$ denotes modulo the sequence $(y_1', \ldots, y_d')\big)$ has the same Hilbert function as $\overline{S}/\overline{I_{m-1}}$. This is only possible if $\overline{I_{m-1}^*}$ is the same power of the maximal ideal as $\overline{I_{m-1}}$. In particular, $\overline{I_{m-1}^*}$ has linear resolution, and therefore I_{m-1}^* has a linear resolution.

Now we show that I_r^* does not have a linear resolution for $r \neq m-1$. Notice that it is enough to show that I_r does not have linear resolution for $r \neq m-1$, since $\beta_{ij}(I_r^*) \geq \beta_{ij}(I_r)$. The a-invariant of the ring S/I_r is equal to $-nr$ i.e. the minimum of the degree of generators of the canonical module of S/I_r is given by nr, see [2, Corollary 1.5]. As the projective dimension of S/I_r is given by $(m-r)(n-r)$[4, Corollary 5.18], we have $M_{(m-r)(n-r)}(S/I_r) = nm - rn$ by (F1) in the first section. Hence $M_{(m-r)(n-r)-1}(I_r) - (m-r)(n-r) + 1 = nm - rn - (m-r)(n-r) + 1 = r(m-r) + 1$ and $M_0(I_r) = r+1$. Hence for $r \neq m-1$, the ideal I_r does not have a linear resolution. □

The Stanley-Reisner ring $S_\sigma = K[\Delta_\sigma]$ being Cohen-Macaulay, admits a graded canonical module ω_σ. In [2], the $a-$invariant of S_σ which is the negative of the least degree of canonical module ω_σ is computed. Next, we want to determine the degree of all the generators of ω_σ for $\sigma = [1, \ldots, r \,|\, 1, \ldots, r]$, $r \leq m-1$. First we need the following lemma:

Lemma 2.6. *Let $\Delta_\sigma = \langle F_1, \ldots, F_t \rangle$ be the simplicial complex with Stanley-Reisner ideal I_σ and $F_1, \ldots, F_t$ be the shelling order Σ. Let $\Delta_i = \langle F_1, \ldots, F_i \rangle$. Then the boundary complex of Δ_i is given by*

$$\partial(\Delta_i) = \big\{G \in \Delta_i : \; F_k \setminus G \not\subset \mathcal{C}(F_k) \text{ for all } \; k \leq i \text{ with } G \subset F_k\big\}.$$

Proof. It is enough to show that the set of facets of $\partial(\Delta_i)$ is given by

$$\mathcal{F}(\partial(\Delta_i)) = \big\{G \in \Delta_i : F_k \setminus G = \{v\}, v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k\big\}.$$

Indeed, if we assume the above statement to be true, then the boundary complex is the set:

$$\big\{H \in \Delta_i : \; H \subset G \text{ for some } G \in \mathcal{F}\big(\partial(\Delta_i)\big)\big\},$$

which is further equal to the set

$$\big\{H \in \Delta_i:\ H \subset G,\ F_k \setminus G = \{v\},\ v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k\big\}.$$

The above set is equal to

$$\big\{H \in \Delta_i:\ F_k \setminus H \not\subset \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } H \subset F_k\big\},$$

as in the statement of the lemma.

Let $\mathcal{S} = \big\{G \in \Delta_i : F_k \setminus G = \{v\}, v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k\big\}$. By Lemma 2.3, we have $\mathcal{S} \subset \mathcal{F}\big(\partial(\Delta_i)\big)$. Now let $G \notin \mathcal{S}$ be of codimension one. It follows that G is of the form $F_k \setminus \{v\}$ where $v \in \mathcal{C}(F_k)$ for some $k \leq i$. Again by Lemma 2.3, there exists $\ell < k$ such that $G \subset F_\ell$. Hence $G = F_\ell \cap F_k$, which implies $G \notin \partial(\Delta_i)$. □

In Theorem 2.4, we have shown that the geometric realization $|\Delta_\sigma|$ of Δ_σ is a ball and therefore the geometric realization $|\partial_\sigma|$ of ∂_σ is a sphere. It is known that simplicial spheres are Gorenstein over any field, see [1, Corollary 5.6.5]. Hence we may apply Theorem 1.1 to compute ω_σ. Before stating the next corollary, we define the notion of a *non-flippable* path. Let D be a path from a to b. Let $v \in D$ such that $\{v+(1,0), v+(0,1)\} \in D$ and neither $v+(1,0)$ nor $v+(0,1)$ is a corner point of D. Then v can be flipped to get a path $D' = (D \setminus \{v\}) \cup \{v+(1,1)\}$. We call such an interchange of the point v to $v+(1,1)$ *a flip*. Notice that the new path D' obtained after a flip from D has the following property: $\mathcal{C}(D) \subset \mathcal{C}(D')$. We call a path D to be a *flippable* path if D could be flipped to get a new path D', otherwise we call D to be a *non-flippable* path. Hence, a non-flippable path D from a to b is a path which has the following property: for all $v \in D$ such that $\{v+(0,1), v+(1,0)\} \subset D$, one has either $v+(0,1)$ or $v+(1,0)$ is a corner point of D. Equivalently, one may notice that a path D from a to b is a non-flippable path if for a path D' from a to b with $\mathcal{C}(D') \supset \mathcal{C}(D)$, one has $D' = D$.

FIGURE 1. A flippable path D and a non-flippable path D' where $D' = (D \setminus \{v\}) \cup \{v'\}$.

We call a facet $F = \bigcup_i C_i$ of the simplicial complex Δ_σ a *non-flippable facet*, if each C_i is a non-flippable path, otherwise we call F a *flippable facet*. Notice that a facet F of Δ_σ is non-flippable if for each facet F' of Δ_σ with $\mathcal{C}(F') \supset \mathcal{C}(F)$, one has $F' = F$. We denote the set of non-flippable facets of Δ_σ by $\mathcal{NF}(\Delta_\sigma)$. Let F, F' be

two facets of Δ_σ with $\mathcal{C}(F) \subset \mathcal{C}(F')$. Then F' is obtained from F by finite number of flips. One has:

Lemma 2.7. *Let F, F' be two facets of Δ_σ, then the following two conditions are equivalent:*

(a) $\mathcal{C}(F) \subset \mathcal{C}(F')$,
(b) $F' \setminus \mathcal{C}(F') \subset F \setminus \mathcal{C}(F)$.

For a given subset Z of $[m] \times [n]$ we denote by X_Z, the monomial $\prod_{(i,j)\in Z} X_{ij}$. We have :

Corollary 2.8. *Let ω_σ be the canonical ideal of $K[\Delta_\sigma]$ and $\mathcal{M}$ denote the set $\{F \setminus\ \mathcal{C}(F) :\ F \in \mathcal{NF}(\Delta_\sigma)\}$. Then the minimal set of generators of ω_σ is given by $G(\omega_\sigma) = \{X_G :\ G \in \mathcal{M}\}$.*

Proof. By Theorem 2.4 and Theorem 1.1, it is enough to show that $\mathcal{M}$ is the set of the minimal inside faces (under inclusion) of Δ_σ.

By Lemma 2.6, we know that the set of inside faces of the simplicial complex Δ_σ is given by $\mathcal{S} = \ \{F \setminus\ Z :\ F \in \mathcal{F}(\Delta_\sigma),\ Z \subset \mathcal{C}(F)\}$. Therefore each minimal inside face G is of the form $F \setminus \mathcal{C}(F)$, $F \in \mathcal{F}(\Delta_\sigma)$.

Let $F \in \mathcal{NF}(\Delta_\sigma)$. Suppose $G = F \setminus \mathcal{C}(F)$ is a not a minimal inside face. Then there exists $G' \subset G$ such that $G' = F' \setminus \mathcal{C}(F')$ is a minimal inside face. By Lemma 2.7, it follows $\mathcal{C}(F') \supset \mathcal{C}(F)$, a contradiction.

Now, let $G = F \setminus \mathcal{C}(F)$ be a minimal inside face. Suppose $F \notin \mathcal{NF}(\Delta_\sigma)$, then there exists a facet F' such that $\mathcal{C}(F') \supset \mathcal{C}(F)$. Again, by Lemma 2.7, it follows then $F' \setminus \mathcal{C}(F') \subset F \setminus \mathcal{C}(F)$, a contradiction. $\square$

In general, to give the explicit expressions of multi-degrees of the generators of canonical ideal ω_σ may not be possible. But we would like to give all possible total degrees of the generators of the canonical ideal ω_σ for $\sigma = [1, \ldots, r\,|\,1, \ldots, r]$, $r \leq m-1$. In this case, I_σ is the ideal generated by all $r+1 \times r+1$ minors of X. For $\sigma = [1, \ldots, r\,|\,1, \ldots, r]$, we denote I_σ by I_r, ω_σ by ω_r and Δ_σ be Δ_r.

From Corollary 2.8, it follows that $|F| - c(F)$, $F \in \mathcal{NF}(\Delta_\sigma)$ are the total degrees of the generators of the canonical ideal ω_σ. We call the corners of the a non-flippable facet $F \in \mathcal{NF}(\Delta_\sigma)$ the *non-flippable corners.* In the case of the simplicial complex Δ_r, we will show that the number t of the non-flippable corners could be any integer between r and $r(m-r)$.

Proposition 2.9. *Let Δ_r be the simplicial complex with the Stanley-Reisner ideal I_r^*. Then there exists a non-flippable facet F of the simplicial complex Δ_r with t corners if and only if $r \leq t \leq r(m-r)$.*

Proof. We will construct a non-flippable facet for any given number of corners between r and $r(m-r)$. As any facet F of Δ_σ is a disjoint union of r paths C_i from (i,n) to (m,i), we notice that the minimum number of non-flippable corner for any path C_i is one and the maximum is $(m-r)$. Hence minimum and maximum number of possible total non-flippable corners are r and $r(m-r)$ respectively. As a path C_i is determined by its corners, we define the non-flippable corners for each path. For

r corners, we define C_i such that $\mathcal{C}(C_i) = (i+1, i+1)$ such that $F = C_1 \cup \cdots \cup C_r$ is a non-flippable facet with r corners; see Figure 2.

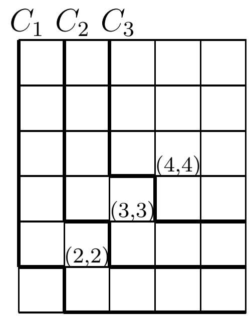

FIGURE 2. A non-flippable facet with $r = 3$ corners.

One can write any $r \leq t \leq r(m-r)$ as $t = r + p(m-r-1) + q$ for $0 \leq p \leq r$ and $0 \leq q < (m-r-1)$. For any such t, we define the corners of the path C_i as follows: For $0 \leq k \leq p-1$, the path C_{r-k} has corners at

$$\big(r-(k-1), n-(k+1)\big), \big(r-(k-2), n-(k+2)\big), \ldots, \big(r-(k-m+r), n-(k+m-r)\big).$$

The path C_{r-p} has corners at

$$(r-p, r-p+q), (r-p+1, r-p+q-1), \ldots, (r-p+q, r-p),$$

and for $1 \leq i \leq r-p-1$, the path C_i has corner at $(i+1, i+1)$. Now $F = \bigcup_{i=1}^{r} C_i$ is a non-flippable facet with exactly $t = r + p(m-r-1) + q$ corners; see Figure 3.

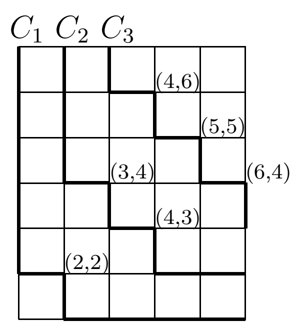

FIGURE 3. A non-flippable facet with $t = r+p(m-r-1)+q$ corners with $m = 6, n = 7, r = 3$ and $p = 1, q = 1$.

Corollary 2.10. *The canonical ideal ω_r has a minimal generator of degree t if and only if $rn \leq t \leq r(n+m-r-1)$.*

Proof. We have $\dim R/I_r = |F| = r(m+n) - r^2$, [4, Corollary 5.18]. Now by Corollary 2.8 and from Proposition 2.9, follows the result. ☐

Next, we want to consider the boundary complex ∂_r of the simplicial complex Δ_r. We want to show that the Stanley-Reisner ring S/I_{∂_r} satisfies the multiplicity conjecture. The geometric realization $|\partial_r|$ of the boundary complex ∂_r is a sphere of dimension $r(m+n)-r^2-1$. Therefore the Stanley-Reisner ring S/I_{∂_r} is a Gorenstein ring, see [1, Corollary 5.6.5]. Hence, the boundary complex ∂_r satisfies properties (P1), (P2), (P3) of Section 1 and by Theorem 1.1, we have $S/I_{\partial_r} = K[\Delta_r]/(\omega_r)$.

Theorem 2.11. *The Stanley-Reisner ring S/I_{∂_r} satisfies the multiplicity conjecture.*

Proof. We need to show that assumptions (A1) and (A2) are satisfied, see Theorem 1.2. As the generators of the canonical ideal ω_r of Δ_r has degrees t where $rn \leq t \leq r(m+n-r-1)$, there exists a minimal inside face of dimension $r(m+n-r-1)-1 = \dim R/I_{\partial_r} - (r+1)$ and there is no inside face of dimension less than $r+1$, see Theorem 1.1. Hence assumption (A1) is satisfied.

For Assumption (A2), we need to show that h-vector of S/I_{∂_r} is unimodal. Let the h-vector of the simplicial complex Δ_r be given by $\left(h_0, \ldots, h_{r(m+n)-r^2}\right)$, then the h-vector $\left(h'_0, \ldots, h'_{r(m+n)-r^2-1}\right)$ of the boundary complex ∂_r is given by (see [18, Page 137]):

$$h_0 - h_{r(m+n)-r^2}, \ldots, h_0 + \cdots + h_{r(m+n)-r^2-1} - h_{r(m+n)-r^2} - \cdots - h_1.$$

By [2, Theorem 2.4] we have that h_i calculates the number of facets F of Δ_r with number of corners $c(F) = i$ and from Corollary 2.9, we get that the maximal number of corners possible are $r(m-r)$, hence $h_t = 0$ for all $r(m-r)+1 \leq t \leq r(m+n)-r^2$. Then it follows that the h-vector of S/I_{∂_r} is given by

$$h'_i = \begin{cases} h'_{r(m+n)-r^2-1-i} = \sum_{j=0}^{i} h_j & \text{for } i = 0, \ldots, r(m-r); \\ \\ \sum_{j=0}^{r(m-r)} h_j & \text{for } j = r(m-r)+1, \ldots, nr-2; \\ \end{cases}$$

Hence h-vector of S/I_{∂_r} is unimodal. □

In the remaining part of this section, we compare the Stanley-Reisner ideal I^*_{m-1} of Δ_{m-1} with its $(I^*_{m-1})^\vee$. We will see in Theorem 2.12 that the dual ideal $(I^*_{m-1})^\vee$ is again the initial ideal of the ideal of the maximal minors of a certain matrix.

Let Δ be a simplicial complex on the vertex set $[n]$ and $I_\Delta \subset K[X_1, \ldots, X_n]$ be the corresponding Stanley-Reisner ideal. There is another simplicial complex $\Delta^\vee$ associated to Δ which is called the *Alexander dual* of Δ. The Alexander dual is defined by the simplicial complex $\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}$. It is easy to see that the complement of the minimal non-faces of the simplicial complex Δ define the facets of the dual complex $\Delta^\vee$ and vise-versa. Hence, the Stanley Reisner ideal $I_{\Delta^\vee}$ is equal to the ideal $\left(X_{i_1} \cdots X_{i_k} : [n] \setminus \{i_1, \ldots, i_k\} \in \mathcal{F}(\Delta)\right)$. One may write $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$ where $P_F = (X_i : i \notin F)$. Therefore the monomials $X_{P_F} = \prod_{X_i \in P_F} X_i$, $F \in \mathcal{F}(\Delta)$ form a set of minimal generators of $I_{\Delta^\vee}$. From here it follows that a monomial g is a minimal generator of $I_{\Delta^\vee}$ if and only if $\mathcal{S} = \{X_i : X_i | g\}$ is a vertex cover of the set of minimal generators $G(I_\Delta)$ of I_Δ (We call a set of indeterminates $\mathcal{S} \subset \{X_1, \ldots, X_n\}$

to be vertex cover of a set of monomials $\{m_1, \ldots, m_k\}$ if for all m_i there exists some $X_j \in S$ such that $X_j | \, m_i$).

Let $X = (X_{ij})$ be a matrix of indeterminates of order $m \times n$. We call a matrix $Y = (Y_{ij})$ of indeterminates of order $(n-m+1) \times n$ a dual of the matrix X if $Y_{i,j+i-1} = X_{j,j+i-1}$ for $i = 1, \ldots, n-m+1$ and $j = 1, \ldots, m$. Notice that if Y is a dual of X, then X is a dual of Y. For example, if

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \end{pmatrix}$$

is a matrix of order 3×4 then a dual matrix Y of order 2×4 can be defined as follows:

$$Y = \begin{pmatrix} X_{11} & X_{22} & X_{33} & Y_{14} \\ Y_{21} & X_{12} & X_{23} & X_{34} \end{pmatrix}.$$

Let again I^*_{m-1} denote the initial ideal of the ideal of maximal minors of an $m \times n$ matrix $X = (X_{ij})$ of indetermiantes and Δ_{m-1} be the simplicial complex with Stanley-Reisner ideal I^*_{m-1}. We denote the Alexander dual of the simplicial complex Δ_{m-1} by $\Delta^{\vee}_{m-1}$ and the corresponding Stanley-Reisner ideal by $(I^*_{m-1})^{\vee}$. Let $Y = (Y_{ij})$ be a dual matrix of X. Let J_{n-m} denote the ideal of the maximal minors of the matrix Y and the initial ideal of J_{n-m} be denoted by J^*_{n-m} (notice J^*_{n-m} does not depend upon the choice of the dual matrix Y). We define a polynomial ring $T = K[X_{ij}, Y_{kj} : \; 1 \le i \le m, 1 \le k \le n-m+1, 1 \le j \le n]$. Then we have:

Theorem 2.12.

$$(I^*_{m-1})^{\vee} T = J^*_{n-m} T.$$

Proof. First we show that the ideal $J^*_{n-m}T$ is contained in the ideal $(I^*_{m-1})^{\vee}T$. Let $g = Y_{1j_1}Y_{2j_2} \cdots Y_{n-m+1,j_{n-m+1}}, \; j_1 < j_2 < \cdots < j_{n-m+1}$ be a minimal generator of the ideal J^*_{n-m}. As $Y_{1j} = X_{jj}, Y_{2j+1} = X_{jj+1}, \ldots, Y_{n-m+1,j+n-m} = X_{jj+n-m}$ for $j = 1, \ldots, m$, the monomial g is of the form $X_{i_1,i_1}X_{i_2,i_2+1} \cdots X_{i_{n-m+1},i_{n-m+1}+n-m}$ for some $1 \le i_1 \le i_2 \le \cdots \le i_{n-m+1} \le m$. We need to show that the set S given by $\{X_{i_1,i_1}, X_{i_2,i_2+1}, \ldots, X_{i_{n-m+1},i_{n-m+1}+n-m}\}$ is a vertex cover for $G(I^*_{m-1})$. Let

$$h = X_{1,1+t_1}X_{2,2+t_2} \cdots X_{m,m+t_m}, \; 0 \le t_1 \le t_2 \le \cdots \le t_m \le n-m$$

be a minimal generator of I^*_{m-1}. We show that there exists $X_{i,j} \in S$ such that $X_{i,j} | \, h$. Suppose the contrary, then $X_{i_k,i_k+(k-1)}$ does not divide h for any $k = 1, \ldots, n-m+1$ which implies $t_{i_k} > k-1$ for $k = 1, \ldots, n-m+1$, in particular $t_{i_{n-m+1}} > n-m$ which is a contradiction.

To show that $(I^*_{m-1})^{\vee}T \subset J^*_{n-m}T$, we need to show that if S is a minimal vertex cover of $G(I^*_{m-1})$, then $\prod_{X_{ij} \in S} X_{ij}$ is a generator of J^*_{n-m}. Since, the monomials $\prod_{i=1}^m X_{i,i+k}, \; k = 0, \ldots, n-m$ are minimal generators of $G(I^*_{m-1})$, we get that the subset of the form $S' = \{X_{i_1,i_1}, X_{i_2,i_2+1}, \ldots, X_{i_{n-m+1},i_{n-m+1}+n-m}\}$ is contained in any minimal vertex cover S of $G(I^*_{m-1})$. Also one may notice that, we must have $1 \le i_1 \le i_2 \le \cdots \le i_{n-m+1} \le m$. Now, the generators of J^*_{n-m} are exactly of the form $\prod_{X_{ij} \in S'} X_{ij}$, hence $(I^*_{m-1})^{\vee}T \subset J^*_{n-m}T$. □

Corollary 2.13. *The Stanley Reisner Ideal I^*_{m-1} has linear quotients.*

Proof. By above theorem and Theorem 2.4 we get that the simplicial complex $\Delta_{m-1}^{\vee}$ gives the triangulation of a shellable linear ball. Now it follows from Theorem 1.4 [12] that I_{m-1}^* has linear quotients. $\square$

3. Polarization of the powers of a maximal ideal

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over the field K and let $\mathfrak{m} = (x_1, \ldots, x_n) \subset S$ denote the maximal graded ideal.

Let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in S. Then the squarefree monomial given by

$$u^P = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij} \in K[x_{11}, \ldots, x_{1a_1}, \ldots, x_{n1}, \ldots, x_{na_n}]$$

is called the *polarization* of u. Let $I = \mathfrak{m}^t$ be the tth power of the maximal ideal. Let $G(I) = \{u_1, \ldots, u_m\}$, then the squarefree monomial ideal $I^P = (u_1^P, \ldots, u_m^P) \subset K[x_{11}, \ldots, x_{1t}, \ldots, x_{n1}, \ldots, x_{nt}]$ is called the *polarization* of I.

Let $\Gamma = \{a \in \mathbb{N}^n \ : \ x^a \notin I\}$ be the multicomplex associated to the ideal I. The detailed information about multicomplexes can be found in [9]. In our case, Γ is a shellable multicomplex, see [9, Theorem 10.5] and all the elements of Γ are its facets. Clearly, Γ consists of those $a \in \mathbb{N}^n$ such that $\sum a(k) \leq t-1$. We define a partial order on the facets of Γ as follows: Let a, b be any two facets of Γ, we say $a < b$ if $\sum_{k=1}^{n} a(k) \leq \sum_{k=1}^{n} b(k)$. This partial order extended to any total order gives us a shelling. We fix a total order and we call the respective shelling Σ. Let $\mathcal{F}(\Gamma) = \{a_1, \ldots, a_m\}$ be the set of the facets of Γ in the shelling order Σ. Let Δ be the simplicial complex with the Stanley-Reisner ideal I^P and let $\mathcal{F}(\Delta)$ be the set of facets of Δ. By [6], it follows that Δ is shellable. Furthermore by [14, Lemma 3.7] and [9, Proposition 10.3] together, it follows that there is a bijection between $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Delta)$ given by

$$\theta : \mathcal{F}(\Gamma) \to \mathcal{F}(\Delta), \ a_k \mapsto F_{a_k}.$$

Here given the facet $a_k = \big(a_k(1), \ldots, a_k(n)\big)$ of Γ, the facet F_{a_k} of Δ is defined to be $\{x_{ij}, \ i = 1, \ldots, n, j = 1, \ldots, t, j \neq a_k(i)+1\}$. Also, $F_{a_1}, \ldots, F_{a_m}$ is a shelling order of the facets of the simplicial complex Δ.

We have the following:

Theorem 3.1. *The geometric realization $|\Delta|$ of the simplicial complex Δ is a shellable linear ball.*

Proof. We already know that $\Delta = \langle F_{a_1}, \ldots, F_{a_m} \rangle$ is a shellable simplicial complex. Note that the Stanley-Reisner ideal $I_\Delta = I^P$ has a linear resolution because the graded Betti numbers of a monomial ideal and its polarization are the same, and $I = \mathfrak{m}^t$ obviously has a linear resolution. Let $\Delta_k = \langle F_{a_1}, \ldots, F_{a_k} \rangle$. We will prove $|\Delta_k|$ is a ball by induction on k as in Theorem2.4. The assertion is obvious for $k = 1$. Assume that $|\Delta_{k-1}|$ is a ball, we will show that $|\Delta_k|$ is a ball where the simplicial complex $\Delta_k = \Delta_{k-1} \cup \langle F_{a_k} \rangle$. Let $\Delta_{k-1} \cap \langle F_{a_k} \rangle = \{G_1, \ldots, G_r\}$ where $G_1, \ldots, G_r$ are codimension one faces of F_{a_k}. By Lemma 2.2, we notice that $|\Delta_k|$ is a ball (assuming that $|\Delta_{k-1}|$ is a ball) if the following two conditions are satisfied:

(1) Each G_ℓ is a subset of exactly one F_{a_i} for $i \le k-1$, which in turn implies that $G_\ell \in \partial\Delta_{k-1}$,
(2) $G_1, \ldots, G_r$ is a proper subset of the boundary complex ∂F_{a_k} of F_{a_k}.

Let $a_k = (s_1, \ldots, s_n)$ where $\sum s_i \le t-1$. Then

$$F_{a_k} = \{x_{ij},\ i = 1, \ldots, n, j = 1, \ldots, t, j \ne s_i + 1\}.$$

Suppose $G_\ell = F_{a_k} \setminus \{x_{i_\ell j_\ell}\}$ where $1 \le i_\ell \le n$ and $1 \le j_\ell \le t$. Then clearly, $G_\ell = F_{a_k} \cap F_{a_{p_\ell}}$ where $a_{p_\ell} = (s_1, \ldots, s_{i_\ell - 1}, j_\ell - 1, s_{i_\ell + 1}, \ldots, s_n)$ and also $G_\ell \not\subset F_{a_q}$ for any $q \le k-1$, $q \ne p_\ell$.

For the second condition, let $1 \le q \le n$ be the minimum integer such that $s_q < t-1$. Let $G = F_{a_k} \setminus \{x_{qt}\}$. Suppose $G \subset F_{a_j}$ for some $j \le k-1$, then it would imply that $a_j = (s_1, \ldots, s_{q-1}, t-1, s_{q+1}, \ldots, s_n)$. Since $\sum a_j(i) \ge t$, we have $a_j \notin \Gamma$, a contradiction. Hence $G \notin \{G_1, \ldots, G_r\}$ and G is a facet of the boundary complex ∂F_{a_k}.

Now by the above theorem and Corollary 1.4, we have the following:

Corollary 3.2. *The simplicial sphere $\partial\Delta$ satisfies the multiplicity conjecture.*

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